The minimum of a multivate polynomial

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If you find any English or math errors or if you have questions and/or suggestions, send me an email at **ilyasssabir7@gmail.com**

The main problem :

Let $a_1,\dots,a_m>0$, $m,q\in\mathbb{N}^\star$ and $n_1,\dots,n_q\in\mathbb{N}^*$, find the maximum value of $K=K(a_1,\dots,a_n)$ such that for all $x_{1,1},\dots,x_{1,m}$ and $x_{2,1},\dots,x_{2,m}$ and $\dots$ and $x_{q,1},\dots,x_{q,m}$ be non-negative real numbers we have

$$\sum_{k=1}^{m} a_k \prod_{j=1}^{q} x_{j,k}^{n_j} \geqslant K \prod_{j=1}^{q}\left(\sum_{k=1}^{m} x_{j,k}\right)^{n_j} \qquad (1)$$

Remark 1.

In the first, we will justify that K existe,
Because of

$$\begin{aligned}\forall j\in[\![1,q]\!],\ (x_{j,1}+\cdots+x_{j,m})^{n_j} &= \sum_{0\leqslant i_1+\cdots+i_m\leqslant n_j}\binom{n_j}{i_1,\dots,i_m}\prod_{l=1}^{m} x_{j,l}^{i_l}\\ &\geqslant x_{j,1}^{n_j}+\cdots+x_{j,m}^{n_j}\end{aligned}$$

We can conclude that

$$\begin{aligned}\prod_{j=1}^{q}\left(\sum_{k=1}^{m} x_{j,k}\right)^{n_j} &\geqslant \prod_{j=1}^{q}(x_{j,1}^{n_j}+\cdots+x_{j,m}^{n_j})\\ &\geqslant \sum_{k=1}^{m}\prod_{j=1}^{q} x_{j,k}^{n_j}\end{aligned}$$

So

$$\sum_{k=1}^{m} a_k \prod_{j=1}^{q} x_{j,k}^{n_j} - \left(\max_{j=1}^{m} a_j\right) \prod_{j=1}^{q} \left(\sum_{k=1}^{m} x_{j,k}\right)^{n_j} \leqslant \sum_{k=1}^{m} a_k \prod_{j=1}^{q} x_{j,k}^{n_j} - \left(\max_{j=1}^{m} a_j\right) \sum_{k=1}^{m} \prod_{j=1}^{q} x_{j,k}^{n_j}$$
$$= \sum_{k=1}^{m} \left(a_k - \max_{j=1}^{m} a_j\right) \prod_{j=1}^{q} x_{j,k}^{n_j}$$
$$\leqslant 0$$

We can remark that the inequality (1) hold for some $K \leqslant \max_{k=1}^{m} a_k$, So the maximum of K existe.

In the next of this article, we try to find the maximum value of $K = K(a_1, \ldots, a_m)$ in some special cases.

Proposition 1. (q=1)

Let $a_1, \ldots, a_m > 0$, $n \in \mathbb{N}^*$. The maximum value of $K = K(a_1, \ldots, a_n)$ such that for all $x_1, \ldots, x_m$ be non-negative real numbers we have

$$\sum_{k=1}^{m} a_k x_k^n \geqslant K \left(\sum_{k=1}^{m} x_k\right)^n \tag{2}$$

is:

$$K = \frac{1}{\left(\frac{1}{\sqrt[n-1]{a_1}} + \cdots + \frac{1}{\sqrt[n-1]{a_m}}\right)^{n-1}}$$

Proof.

Let $x_1, \ldots, x_m > 0$

And let $y_1, \ldots, y_m$ be non-negative real numbers such that $y_1 + \cdots + y_m = x_1 + \cdots + x_m$

According to **Hölder's inequality** we have

$$\left(\sum_{k=1}^{m} a_k x_k^n\right) \left(\sum_{k=1}^{m} a_k y_k^n\right)^{n-1} \geqslant \left(\sum_{k=1}^{m} a_k x_k y_k^{n-1}\right)^n \tag{3}$$

So,

$$\sum_{k=1}^{m} a_k x_k^n \geqslant \frac{\left(\sum_{k=1}^{m} a_k x_k y_k^{n-1}\right)^n}{\left(\sum_{k=1}^{m} a_k y_k^n\right)^{n-1}} \tag{4}$$

We will choose $y_1, \ldots, y_m$ such that $a_1 y_1^{n-1} = \cdots = a_m y_m^{n-1} = C$, and if then

$$\sum_{k=1}^{m} a_k x_k^n \geqslant \frac{C^n \left(\sum_{k=1}^{m} x_k\right)^n}{C^{n-1} \left(\sum_{k=1}^{m} y_k\right)^{n-1}} = C \sum_{k=1}^{m} x_{1k} \tag{5}$$

We have

$$y_k = \sqrt[n-1]{\frac{C}{a_k}}, \text{for all } k = 1, \ldots, m$$

Therefore

$$\sqrt[n-1]{C}\left(\frac{1}{\sqrt[n-1]{a_1}} + \cdots + \frac{1}{\sqrt[n-1]{a_m}}\right) = x_1 + \cdots + x_m$$

So,

$$C = \frac{(x_1 + \cdots + x_m)^{n-1}}{\left(\frac{1}{\sqrt[n-1]{a_1}} + \cdots + \frac{1}{\sqrt[n-1]{a_m}}\right)^{n-1}}$$

We conclude that

$$\sum_{k=1}^{m} a_k x_k^n \geqslant \frac{(x_1 + \cdots + x_m)^n}{\left(\frac{1}{\sqrt[n-1]{a_1}} + \cdots + \frac{1}{\sqrt[n-1]{a_m}}\right)^{n-1}}$$

And the equality gold for $x_k = \frac{1}{\sqrt[n-1]{a_k}\left(\frac{1}{\sqrt[n-1]{a_1}} + \cdots + \frac{1}{\sqrt[n-1]{a_m}}\right)}$, $k = 1, \ldots, m$

The proof is completed. □

Proposition 2.

Let $a_1, \ldots, a_m > 0$, $m, q \in \mathbb{N}^\star$ and $n_1, \ldots, n_q \in \mathbb{N}^*$, The maximum value of $K = K(a_1, \ldots, a_n)$ such that for all $x_{1,1}, \ldots, x_{1,m}$ and $x_{2,1}, \ldots, x_{2,m}$ and … and $x_{q,1}, \ldots, x_{q,m}$ be non-negative real numbers verify for all $(j, l, k) \in [\![1, q]\!]^2 \times [\![1, m]\!]$

$$\frac{x_{j,k}}{x_{j,1} + \cdots + x_{j,m}} = \frac{x_{l,k}}{x_{l,1} + \cdots + x_{l,m}}$$

we have

$$\sum_{k=1}^{m} a_k \prod_{j=1}^{q} x_{j,k}^{n_j} \geqslant K \prod_{j=1}^{q}\left(\sum_{k=1}^{m} x_{j,k}\right)^{n_j} \tag{6}$$

is

$$\boldsymbol{K = \frac{1}{\left(\frac{1}{\sqrt[n-1]{a_1}} + \cdots + \frac{1}{\sqrt[n-1]{a_m}}\right)^{n-1}}}$$

Where $n = n_1 + \cdots + n_q$

Proof.

We have for all $(j, l, k) \in [\![1, q]\!]^2 \times [\![1, m]\!]$

$$\frac{x_{j,k}^{n_j}}{(x_{j,1} + \cdots + x_{j,m})^{n_j}} = \frac{x_{l,k}^{n_j}}{(x_{l,1} + \cdots + x_{l,m})^{n_j}}$$

So,

$$\prod_{j=1}^{q}\frac{x_{j,k}^{n_j}}{\left(x_{j,1}+\cdots+x_{j,m}\right)^{n_j}} = \prod_{j=1}^{q}\frac{x_{l,k}^{n_j}}{\left(x_{l,1}+\cdots+x_{l,m}\right)^{n_j}} = \frac{x_{l,k}^{\sum\limits_{j=1}^{q} n_j}}{\left(x_{l,1}+\cdots+x_{l,m}\right)^{\sum\limits_{j=1}^{q} n_j}} \tag{7}$$

By (7), we get

$$\begin{aligned}\sum_{k=1}^{m} a_k \prod_{j=1}^{q} x_{j,k}^{n_j} &= \frac{\prod\limits_{j=1}^{q}\left(x_{j,1}+\cdots+x_{j,m}\right)^{n_j}}{\left(x_{1,1}+\cdots+x_{1,m}\right)^{\sum\limits_{j=1}^{q} n_j}}\sum_{k=1}^{m} a_k x_{1,k}^{\sum\limits_{j=1}^{q} n_j}\\ &= \frac{\prod\limits_{j=1}^{q}\left(x_{j,1}+\cdots+x_{j,m}\right)^{n_j}}{\left(x_{1,1}+\cdots+x_{1,m}\right)^{n}}\sum_{k=1}^{m} a_k x_{1,k}^{n}\end{aligned}$$

According to the proposition 1, we have

$$\sum_{k=1}^{m} a_k x_{1,k}^{n} \geqslant \frac{\left(x_{1,1}+\cdots+x_{1,m}\right)^{n}}{\left(\frac{1}{\sqrt[n-1]{a_1}}+\cdots+\frac{1}{\sqrt[n-1]{a_m}}\right)^{n-1}} \tag{8}$$

We conclude that

$$\sum_{k=1}^{m} a_k \prod_{j=1}^{q} x_{j,k}^{n_j} \geqslant \frac{1}{\left(\frac{1}{\sqrt[n-1]{a_1}}+\cdots+\frac{1}{\sqrt[n-1]{a_m}}\right)^{n-1}}\prod_{j=1}^{q}\left(\sum_{k=1}^{m} x_{j,k}\right)^{n_j}$$

And the equality gold for $x_k = \frac{1}{\sqrt[n-1]{a_k}\left(\frac{1}{\sqrt[n-1]{a_1}}+\cdots+\frac{1}{\sqrt[n-1]{a_m}}\right)}$, $k=1,\ldots,m$

The proof is completed □